Mathematical supplement to:

Self-organization of treadmilling microtubules into a polar array

by Ivan V. Maly and Gary G. Borisy

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Consider an arbitrary chord AB crossing the cytoplasmic domain. Let us introduce the co-ordinate axis x on which the chord AB lies, such that the coordinate of the point A is x = 0 and the coordinate of the point B is x = L > 0. We shall consider microtubules that geometrically belong to the chord AB and whose direction, from the minus-ends to the plus-ends, coincides with the direction of the x-axis. Such microtubules will be called AB microtubules. We will also say that a microtubule is in state 1 if its plus-end is growing and that it is in state 2 if its plus-end is paused at the plasma membrane. Let m₁(t, xₘ, xₚ) be the concentration, at time t, of AB microtubules in state 1, such that their minus-ends are located at x = xₘ and their plus-ends are located at x = xₚ. Similarly, let m₂(t, xₘ) be the concentration, at time t, of AB microtubules in state 2, such that their minus-ends are located at x = xₘ. The rate of plus-end growth will be denoted as vₑ, the rate of minus-end shortening as vₘ, and the rate of microtubule nucleation per unit area of the cytoplasmic domain as n. Now the dynamics of the AB microtubules can be formulated in mathematical terms so that their consequence for the array polarity can be derived. The rate of change in the distribution of the AB microtubules in state 1 is:

\[
\frac{\partial}{\partial t} m_1(t, x_m, x_p) = -v_p \frac{\partial}{\partial x_p} m_1(t, x_m, x_p) - v_m \frac{\partial}{\partial x_m} m_1(t, x_m, x_p)
\]

[Eqn 1]

Similarly, for state 2:

\[
\frac{\partial}{\partial t} m_2(t, x_m) = v_p m_1(t, x_m, L) - v_m \frac{\partial}{\partial x_m} m_2(t, x_m)
\]

[Eqn 2]

The fact that, at nucleation, the plus- and minus-ends of a microtubule are in the same position in the cell is represented as a boundary condition:

\[
m_1(t, x_m, x_p) \bigg|_{x_p = 0} = \frac{n}{2\pi(v_p - v_m)}
\]

[Eqn 3]

Another boundary condition represents the fact that no microtubule ends enter the cell from beyond its boundaries:

\[
m_1(t, 0, x_p) \bigg|_{x_p = 0} = 0, \quad m_2(t, 0) = 0
\]

[Eqn 4]

It can be verified by substitution that the following forms of the functions m are the time-constant solution to the above differential equations and satisfy the specified boundary conditions, so they describe the steady-state distribution of the AB microtubules:

\[
m_1(x_m, x_p) \bigg|_{x_m = x_p} = \frac{n}{2\pi(v_p - v_m)}, \quad m_2(x_m, x_p) \bigg|_{x_m = x_p} = \frac{n}{2\pi(v_p - v_m)} \left( \frac{v_p}{v_m} x_m - L \right)
\]

[Eqn 5]

AB microtubules whose minus- and plus-ends are at different sides of a certain position, x, contribute to the steady-state concentration of microtubules at that position regardless of their
state or the exact position of their ends. Hence, the integration over the positions of ends and summation over the states yields the steady-state concentration of \( AB \) microtubules crossing the position with the coordinate \( x \):

\[
f(x) = \int_0^l m_1(x_1, x) dx_1 dx + \int_0^l m_2(x_2, x) dx_2 = bx^2, \quad b = \frac{(v_p - v_m)}{4\pi v_p v_m}
\]  

[Eqn 6]

Now consider a point \( M \) that belongs to the chord \( AB \) and is located at the distance \( r \) from the center of the flat, round domain of cytoplasm. Let \( \phi \) be the angle of the \( x \)-axis with respect to the radius of the domain that is drawn through the point \( M \). If the radius of the domain is \( R \), then the coordinate \( x \) of the point \( M \) is:

\[
x(r, \phi) = \sqrt{R^2 - (r \sin \phi)^2} + r \cos \phi
\]  

[Eqn 7]

Using this expression for \( x \), we can express \( f \) as a function of \( r \) and \( \phi \):

\[
f(r, \phi) = b\left(\sqrt{R^2 - (r \sin \phi)^2} + r \cos \phi\right)^2
\]  

[Eqn 8]

This function gives the concentration of the microtubules that pass through a certain point at the distance \( r \) from the center of the flat, round cytoplasmic domain in the direction, from minus- to plus-ends, that differs from the radial direction by the angle \( \phi \).